AUTOMATIC GENERATION OF PROGRAMS THAT JOINTLY OPTIMIZE CHARACTERISTICS OF ANALOG FILTER DESIGNS

Brian L. Evans[1], Douglas R. Firth[2], Kennard D. White[3], and Edward A. Lee[1]

[1] *Dept. of EECS, University of California, Berkeley, CA, 94720-1770*
[2] *Precision Filters, Inc., 240 Cherry Street, Ithaca, NY 14850*
[3] *Diva Communications, Berkeley, CA, 94720*

Abstract - **This paper gives an algebraic framework for designing analog filters that are jointly optimized for magnitude, phase, and step responses, and filter quality. We formulate the design problem as a sequential quadratic programming (SQP) problem and use symbolic mathematical software to translate the SQP formulation into working** MATLAB **programs to optimize analog filters.**

1. INTRODUCTION

This paper derives an extensible framework for jointly optimizing the behavior of analog filters in terms of their magnitude, phase, and step responses, and their implementation in terms of their quality factors. The optimization is performed with respect to the pole-zero locations, subject to constraints on magnitude response, quality factors, and peak overshoot. We formulate the procedure at an algebraic level, and use symbolic mathematical software to generate the code to compute the optimization.

We convert the constrained non-linear optimization filter design problem to a sequential quadratic programming (SQP) problem. SQP requires that the objective function [1] and the constraints [2] be real-valued and twice continuously differentiable with respect to the free parameters. SQP relies on the gradients of the objective function and constraints. SQP methods have been previously applied to optimizing loss and delay in digital filter designs [3].

Section 2 reviews notation. Section 3 derives a family of weighted, differentiable objective functions to measure the deviation in magnitude response, deviation in linear phase response, filter quality, and peak overshoot of the step response, of an analog filter. In the derivation, we find a new analytic approximation for the peak overshoot. Section 4 converts filter specifications into differentiable constraints. Section 5 gives an example of an optimized filter design.

We bring the equations for the objective function and constraints together in the symbolic mathematics environment *Mathematica*, and program it to

(1) compute the gradients of the objective function and constraints symbolically,

(2) convert the objective function, constraints, and their gradients into MATLAB [4] functions,

(3) generate a MATLAB script to run the numerical optimization.

When the designer changes the objective function, our symbolic software will then recompute the gradients and regenerate the numerical optimization code. In essence, we have bridged the gap between the symbolic work designers often do on paper and the working computer implementation. We have eliminated algebraic errors in hand calculation and bugs in coding the software implementation.

2. NOTATION

We represent an analog filter by its n complex conjugate pole pairs $p_k = a_k \pm j b_k$ where $a_k < 0$ and its r complex conjugate zero pairs $z_l = c_l \pm j d_l$ where $c_l < 0$, such that $r < n$. The magnitude and unwrapped phase responses of an all-pole filter, expressed as real-valued differentiable functions, are

$$\begin{aligned} |H(j\omega)| &= \prod_{k=1}^{n} \frac{\sqrt{a_k^2 + b_k^2}}{\sqrt{a_k^2 + (\omega + b_k)^2}\sqrt{a_k^2 + (\omega - b_k)^2}} \\ &= \prod_{k=1}^{n} \frac{\sqrt{a_k^2 + b_k^2}}{\sqrt{(\omega^2 + 2(a_k^2 - b_k^2))\omega^2 + (a_k^2 + b_k^2)^2}} \end{aligned} \quad (1)$$

$$\angle H(j\omega) = \sum_{k=1}^{n} \arctan\left(\frac{\omega - b_k}{a_k}\right) + \arctan\left(\frac{\omega + b_k}{a_k}\right) \quad (2)$$

We factor the polynomial under the square root in (1) into Horner's form because it has better numerical properties. Together with the zero pairs, the magnitude and unwrapped phase responses are

$$|G(j\omega)| = |H(j\omega)| \cdot \prod_{l=1}^{r} \frac{\sqrt{(\omega^2 + 2(c_l^2 - d_l^2))\omega^2 + (c_l^2 + d_l^2)^2}}{\sqrt{c_l^2 + d_l^2}} \quad (3)$$

$$\angle G(j\omega) = \angle H(j\omega) - \sum_{l=1}^{r} \arctan\left(\frac{\omega - d_l}{c_l}\right) + \arctan\left(\frac{\omega + d_l}{c_l}\right) \quad (4)$$

In this paper, Q represents quality factors, ϵ represents a small positive number, σ denotes deviation, m represents slope of a line, t is time, and W is a weighting factor.

3. OBJECTIVE FUNCTIONS

In this section, we derive measures of closeness to an ideal magnitude and phase response, quality factors, and peak overshoot. The objective function is a weighted combination of these measures, and a non-negative function.

3.1. Deviation in the Magnitude Response

We measure the deviation from the ideal in terms of magnitude response in the passband, transition bands, and stopbands separately. Based on the notation in Figure 1, the five components of the objective function relating to the deviation from an ideal magnitude response in the least squares sense are:

$$\sigma_{sb1} = \int_{0}^{\omega_{s1}} F_{s1}(\omega) |H(j\omega)|^2 \, d\omega \quad (5)$$

$$\sigma_{tb1} = \int_{\omega_{s1}}^{\omega_{p1}} F_{t1}(\omega) \left(|H(j\omega)| - (m_1\,\omega - m_1\,\omega_{s1})\right)^2 \, d\omega \quad (6)$$

$$\sigma_{pb} = \int_{\omega_{p1}}^{\omega_{p2}} F_{p}(\omega) \left(|H(j\omega)| - 1\right)^2 \, d\omega \quad (7)$$

$$\sigma_{tb2} = \int_{\omega_{p2}}^{\omega_{s2}} F_{t2}(\omega) \left(|H(j\omega)| - (m_2\,\omega - m_2\,\omega_{s2})\right)^2 \, d\omega \quad (8)$$

$$\sigma_{sb2} = \int_{\omega_{s2}}^{\infty} F_{s2}(\omega) |H(j\omega)|^2 \, d\omega \quad (9)$$

where $F_p(\omega)$, $F_{t1}(\omega)$, $F_{t2}(\omega)$, and $F_s(\omega)$ are integrable weighting functions, and m_1 and m_2 are the slopes of the ideal response in the transition regions defined as $m_1 = 1/(\omega_{p1} - \omega_{s1})$ and $m_2 = 1/(\omega_{p2} - \omega_{s2})$.

3.2. Deviation in the Phase Response

For the passband response, the objective function measures the deviation from linear phase over some range of frequencies (usually over the passband):

$$\sigma_{phase} = \int_{\omega_1}^{\omega_2} \left(\angle H(j\omega) - m_{lp}\omega\right)^2 \, d\omega \quad (10)$$

where m_{lp} is the ideal slope of the linear phase response. Unfortunately, one does not know the value

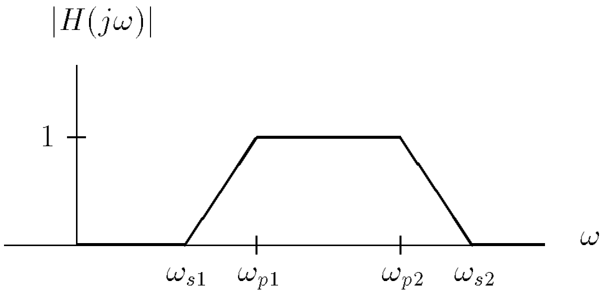


Figure 1: The Ideal Magnitude Response

of m_{lp} *à priori*. We can compute it as the slope of the line in ω that minimizes (10):

$$\min_{m_{lp}} \int_{\omega_1}^{\omega_2} \left(\angle H(j\omega) - m_{lp}\omega\right)^2 \, d\omega \quad (11)$$

In (11), the $H(j\omega)$ term does not depend on m_{lp}, so the integrand is quadratic in m_{lp}. To find the minimum, we take the derivative with respect to m_{lp}, set it to zero, and solve for m_{lp}:

$$m_{lp} = \frac{\int_{\omega_1}^{\omega_2} \angle H(j\omega)\,\omega\,d\omega}{\int_{\omega_1}^{\omega_2} \omega^2\,d\omega} \quad (12)$$

After evaluating the integrals,

$$m_{lp} = \frac{3}{2(\omega_2^3 - \omega_1^3)} \sum_{k=1}^{n} \left[\, f_{lp}(\omega_2) - f_{lp}(\omega_1) \,\right] \quad (13)$$

where for the all-pole case $f_{lp}(\omega)$ is

$$f_{lp}(\omega) = 2\omega a_k + (b_k^2 - a_k^2 - \omega^2)\cdot \left(\arctan\left(\frac{\omega - b_k}{a_k}\right) + \arctan\left(\frac{\omega + b_k}{a_k}\right)\right) + a_k b_k \left(\log\left(1 + \frac{(\omega - b_k)^2}{a_k^2}\right) - \log\left(1 + \frac{(\omega + b_k)^2}{a_k^2}\right)\right)$$

Using *Mathematica*, we computed the definite integrals in (12) and verified the answers. Now that we have a closed-form solution for m_{lp}, we can substitute (13) into (10) to obtain a rather complicated but differentiable expression for the deviation from linear phase.

3.3. Filter Quality

The quality factor measures the relative distance of poles from the imaginary frequency axis. The lower the quality factor, the less likely that the pole will cause oscillations in the output (e.g., as a response to noisy input). The quality factor Q_k for the kth second-order section with conjugate poles $a_k \pm j b_k$ (with $a_k < 0$) and the effective overall quality factor Q_{eff} are

$$Q_k = \frac{\sqrt{a_k^2 + b_k^2}}{-2a_k} \qquad Q_{\text{eff}} = \left(\prod_{k=1}^{n} Q_i\right)^{\frac{1}{n}} \quad (14)$$

where $Q_k, Q_{\text{eff}} \geq 0.5$. $Q_k = 0.5$ corresponds to a double real-valued pole ($b_k = 0$), and $Q_k = \infty$ corresponds to an ideal oscillator ($a_k = 0$). We define Q_{eff} as the geometric mean of the quality factors, and use $Q_{\text{eff}} - 0.5$ to measure the filter quality.

3.4. Peak Overshoot in the Step Response

A closed-form solution for the overshoot of a second-order analog filter exists, but one for higher-order filters does not. From the step response, we can numerically compute the peak overshoot and the time t_{peak} at which it occurs. In order to make the peak overshoot calculation differentiable, this section derives an analytic expression that approximates t_{peak} in terms of the pole-zero locations. This derivation assumes that there are no multiple poles, so some classes of filters are excluded [5].

The Laplace transform of the step response is

$$\frac{H(s)}{s} = \frac{1}{s}\left[\prod_{k=1}^{n} \frac{\sqrt{a_k^2 + b_k^2}}{s^2 - 2a_k s + a_k^2 + b_k^2}\right] \tag{15}$$

Assuming no duplicate poles, partial fractions yields

$$\frac{H(s)}{s} = \frac{1}{s}\left[\sum_{k=1}^{n} \frac{C_k s + D_k}{s^2 - 2a_k s + a_k^2 + b_k^2}\right] \tag{16}$$

$$\begin{aligned} C_k &= 2|A_k|\cos(\angle A_k) \\ D_k &= -2|A_k|\left(a_k \cos(\angle A_k) + b_k \sin(\angle A_k)\right) \\ A_k &= \left[H(s)(s - p_k)\right]_{s=p_k} = |A_k| e^{j\angle A_k} \end{aligned}$$

$|A_k|$ and $\angle A_k$ can be expressed as real-valued differentiable functions of the pole and zero locations. After inverse transforming (16), the overall step response is

$$h_{\text{step}}(t) = \sum_{k=1}^{n} \frac{D_k}{a_k^2 + b_k^2}\left(1 - e^{a_k t}\left(\cos(b_k t) - \frac{a_k + \gamma_k}{b_k}\sin(b_k t)\right)\right) \tag{17}$$

where $\gamma_k = C_k\left(a_k^2 + b_k^2\right)/D_k$. The $D_k/\left(a_k^2 + b_k^2\right)$ term, which can be positive or negative, is the steady-state value for that second-order section.

For a second-order section, the time at which the maximum overshoot occurs is

$$t_{\text{peak}}^k = -\frac{1}{b_k}\left(\arctan\left(\frac{\gamma_k b_k}{a_k^2 + \gamma_k a_k + b_k^2}\right) + \pi\right) \tag{18}$$

and this reduces in the all-pole case to

$$t_{\text{peak}}^k = -\frac{1}{b_k}\left(\arctan\left(\angle A_k\right) + \frac{\pi}{2}\right) \tag{19}$$

We construct the following function to approximate t_{peak} for the purposes of computing derivates:

$$t_{\text{peak}} \approx \frac{1}{n}\sum_{k=1}^{n} t_{\text{peak}}^k \Rightarrow t_{\text{peak}} = \beta\,\frac{1}{n}\sum_{k=1}^{n} t_{\text{peak}}^k \tag{20}$$

where β is set to the true value of t_{peak} (found numerically) divided by the approximation $\frac{1}{n}\sum_{k=1}^{n} t_{\text{peak}}^k$. We verified (20) using the SQP routine. We measure the peak overshoot cost by using $\left(h_{\text{step}}(t_{\text{peak}}) - 1\right)^2$.

4. CONSTRAINTS

This section discusses two sets of constraints. The first specifies the magnitude response, quality, and peak overshoot, and the second prevents numerical instabilities in the computations. We sample the magnitude response at a set of passband frequencies $\{w_m\}$ and stopband frequencies $\{w_l\}$:

$$1 - \delta_p \leq |H(jw_m)| \leq 1 \quad \text{and} \quad |H(jw_l)| \leq \delta_s \tag{21}$$

We compute the maximum overshoot by finding the maximum value of step response in (17) by searching over $t \in [\min_k t_{\text{peak}}^k, \max_k t_{\text{peak}}^k]$. Before finding the gradient of this constraint, we substitute the analytic approximation for t_{peak}, given by (20), into (17).

When the analog filter is implemented, the second-order sections will be cascaded in order of ascending quality factors. This arrangement minimizes the oscillatory behavior of the final sections because the earlier sections will have attenuated the input signal. Nonetheless, an upper limit Q_{max} does exist in practice on the quality factors. We set Q_{max} to 10 for $\omega_{p2} < 2\pi(10)$ kHz, and 25 otherwise:

$$\frac{\sqrt{a_k^2 + b_k^2}}{-2a_k} < Q_{max} \quad \text{for} \quad k = 1 \ldots n \tag{22}$$

The real components of the poles and zeroes appear in the denominator of the phase response. Therefore, we constrain the real parts of the poles and zeroes to be a neighborhood away from 0.

$$\begin{aligned} a_k < -\epsilon_{div} < 0 \quad &\text{for} \quad k = 1 \ldots n \\ c_l < -\epsilon_{div} < 0 \quad &\text{for} \quad l = 1 \ldots r \end{aligned}$$

Ensuring the numerical stability of the denominators of $|A_k|$ and $\angle A_k$ in (16) requires that

$$\sqrt{a_k - a_m} > \epsilon_{div} \quad \text{for } k = 1 \ldots n \text{ and } m = k+1 \ldots n$$

These constraints are analogous to preventing duplicate poles and poles spaced too closely to one another.

5. AN EXAMPLE FILTER DESIGN

An all-pole lowpass filter will be minimized in terms of its overshoot and deviation from linear phase. The specifications on the magnitude response are $w_p = 20$ rad/sec with $\delta_p = 0.21$ and $w_s = 30$ rad/sec with $\delta_s = 0.31$. We use a fourth-order Butterworth filter with poles at $-8.4149 \pm j20.3153$ and $-20.3153 \pm j8.4149$.

In the objective function, we weight the linear phase cost by 0.1 and overshoot cost by 1. The initial value of the objective function is 1.17, which almost entirely comes from overshoot cost. 97% of final value of the objective function, 4.7×10^{-5}, is the cost of the deviation from linear phase. The final poles, in order of ascending quality factors, are $-19.5623 \pm j0.6255$ and $-7.7918 \pm j22.8984$. For these poles $a_1 \pm jb_1$ and $a_2 \pm jb_2$, the gradient with respect to $\{a_1, b_1, a_2, b_2\}$ is $\{-2.251 \times 10^{-5}, -5.507 \times 10^{-6}, 3.116 \times 10^{-5}, 4.179 \times 10^{-5}\}$. Since the second filter section is more sensitive than the first filter section with respect to perturbations in the pole locations, the second filter section should be implemented with better components.

Figure 2 plots the frequency and step responses for the initial and final filters. The figures illustrates that the optimization procedure effectively trades off magnitude response in the passband for a more linear phase response in the passband and a lower overshoot.

6. CONCLUSION

We give an extensible algebraic framework for optimizing analog filter designs in terms of magnitude response, phase response, quality, and peak overshoot. We formulate the constrained non-linear optimization problem as a SQP problem, and derive differentiable, real-valued, constraints and non-negative objective functions. We give a filter design example.

We use *Mathematica* to bring the equations for the objective functions and constraints together. The key idea is that our approach eliminates errors in hand calculations and errors in programming. It also allows a researcher to create new filter design programs by simply redefining the objective function, and our software will take care of recomputing the gradients and regenerating the source code.

7. REFERENCES

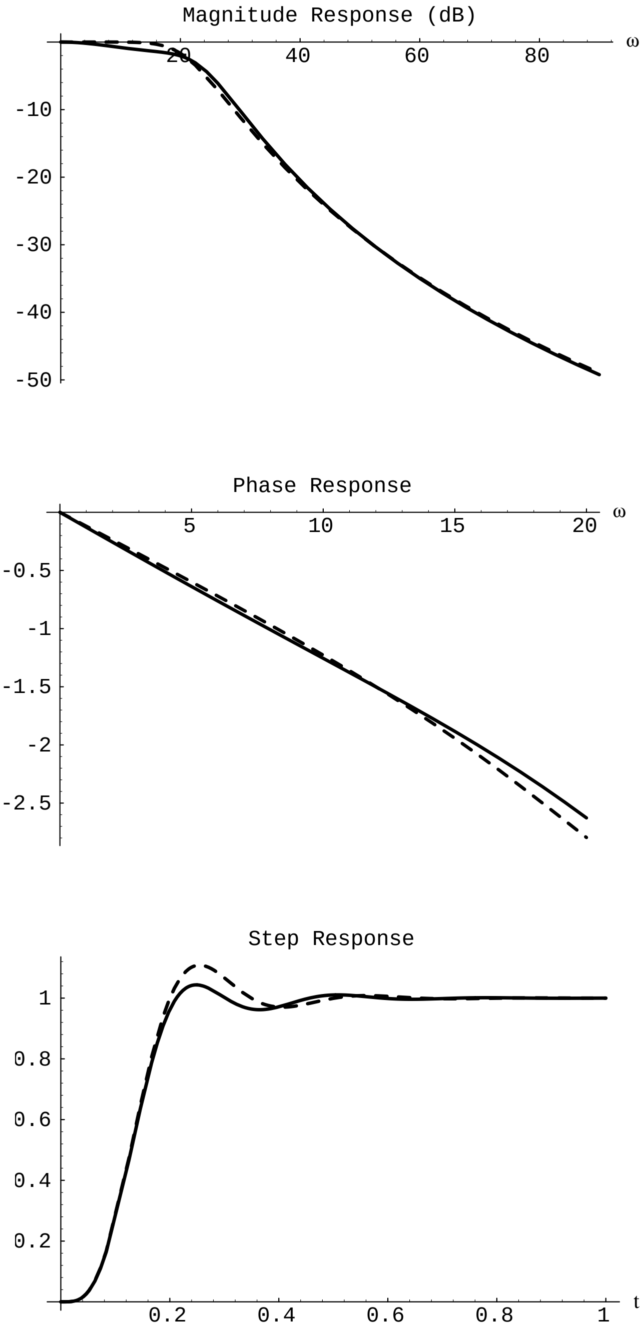
[1] S. Wright, "Convergence of SQP-like methods for constrained optimization," *SIAM Journal on Control and Optimization*, vol. 27, pp. 13–26, Jan. 1989.

[2] K. Schittkowski, "NLPQL: A Fortran subroutine solving constrained nonlinear programming problems," *Annals of Operations Research*, vol. 5, no. 1-4, pp. 485–500, 1986.

[3] S. Lawson and T. Wicks, "Improved design of digital filters satisfying a combined loss and delay specification," *IEE Proceedings G: Circuits, Devices and Systems*, vol. 140, pp. 223–229, June 1993.

[4] C. Moler, J. Little, and S. Bangert, *Matlab User's Guide*. Natick, MA: The MathWorks Inc., 1989.

[5] M. Biey, S. Coco, and A. Premoli, "Maximally-flat rational approximation with multiple critical pole-pair," *IEE Proceedings G*, vol. 138, pp. 244–252, Apr. 1991.



$w_p = 20$ rad/s, $\delta_p = 0.21$, $w_s = 30$ rad/s, $\delta_s = 0.31$

- - - initial filter —— optimized filter

We are trading linear phase response and peak overshoot for magnitude response while keeping the magnitude response within specification.

Figure 2: Fourth-Order Lowpass Filter with Optimized Phase and Step Responses