INTERACTIVE GRAPHICAL DESIGN OF TWO-DIMENSIONAL COMPRESSION SYSTEMS

*Brian L. Evans**

Dept. of Electrical Engineering
and Computer Sciences
Univ. of California at Berkeley
Berkeley, CA 94720 USA
ble@eecs.berkeley.edu

F. Ayhan Sakarya

Elektrik-Elektronik Fakültesi
Yıldız Teknik Üniversitesi
İstanbul, Türkiye
sakarya@tryildiz.bitnet

Abstract

The paper gives an automated procedure to design rational decimation compression systems that resample two-dimensional bandpass signals at their Nyquist rates. The procedure takes a sketch of the passband in the frequency domain, circumscribes it with a parallelogram, and *linearly* maps the parallelogram onto one period of the frequency domain. Thus, the compression system only has linear components.

1 INTRODUCTION

Two-dimensional decimation systems can be used to reduce the amount of data for applications in which the crucial data occupies a certain frequency band. For example, images are often oversampled, so most of the signal energy resides at low frequencies. If the high-frequency content (edges, texture, etc.) is not important, then the image can be decimated to a lower spatial resolution. In video processing, two-dimensional decimators can be used to convert sequences of images from interlaced to non-interlaced format. These decimators preserve the frequency content in a diamond-shaped (i.e., parallelogram-shaped) passband centered at zero frequency. Seismic data is sampled in position and time [1]. Data falling on the same position-time line corresponds to waves having the same velocity. Fan filters are used to pass ranges of velocities. When the range of passed velocities is small, the fan filters produce "narrowband" signals. The narrowband signals take the shape of a parallelogram in the frequency domain if the passband includes either frequency axis.

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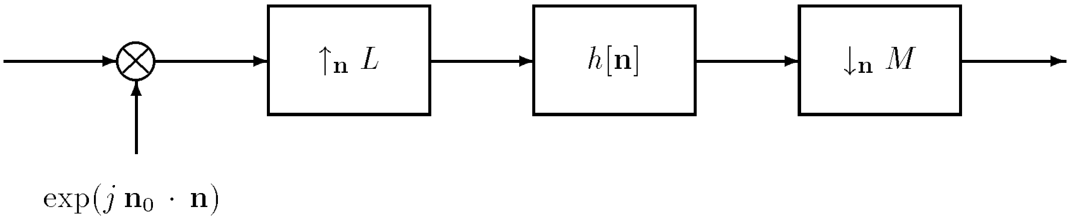


Figure 1: Flow Graph of a Two-Dimensional Decimator

The paper discusses the automation of the design of compression systems that resample two-dimensional bandpass signals at their Nyquist rate. The bandpass signal is represented by a polygon. The engineer can sketch the polygon with a mouse or specify its vertices with mathematical formulas. From the polygonal passband, the automated procedure will compute the parameters for the compression system shown in Figure 1. The entire procedure is encoded in the *Mathematica* [2] symbolic mathematics environment. Sketching the passband utilizes the notebook interface to *Mathematica*, and the conversion from the polygonal passband to a decimation system is handled by the signal processing packages [3, 4] for *Mathematica*.

2 THEORY FOR DECIMATOR DESIGN

This section discusses the theory underlying rational decimation systems. Rational decimation systems extract a connected portion of the frequency domain (the passband) and resample it at a lower rate. To resample the desired passband at the Nyquist rate, the passband is shifted down to baseband (DC) and mapped onto one period of the frequency domain while preventing aliasing and imaging effects. The period is chosen to be the fundamental frequency tile $\omega_1 \in [-\pi, \pi) \bigcup \omega_2 \in [-\pi, \pi)$. Using this period, the steps involved in designing a two-dimensional decimator amount to

1. computing the minimal rectangle with floating-point coordinates that circumscribes the passband,

2. finding the parallelogram whose coordinates are rational multiples of π, whose area is minimal, and whose extent includes the minimal rectangle,

3. shifting the center of the parallelogram to the origin (baseband),

4. defining the rational matrix H that maps the parallelogram onto the fundamental frequency tile, and

5. factoring the rational matrix H into the two integer matrices L and M.

Step 1 of the design procedure finds the rectangle of minimal area that circumscribes the passband. For each polygon edge of the passband, the polygon is rotated so that the edge would lie on the horizontal axis. The rectangle of smallest area that circumscribes the rotated polygon is then computed by finding the minimum and maximum coordinates of the rotated vertices. After all of the edges have been processed, the rectangle with the smallest area is chosen. Then, the rectangle is rotated to the location of the passband.

From the rectangle computed in Step 1, Step 2 finds the parallelogram with minimal area that circumscribes the rectangle. Unlike the coordinates of the rectangle's vertices, each coordinate of the parallelogram's vertices must be a rational number times π. The procedure first sorts the vertices of the rectangle such that the first vertex is the upper left corner and the other vertices are in clockwise order. Then, the procedure finds the parallelogram that circumscribes the rectangle by (a) rationalizing the division of three of the four rectangle coordinates with π so that the new coordinates are "outside" the rectangle, and (b) computing the fourth coordinate from the other three so as to make sure that the bounding region is a parallelogram.

Step 3 only requires averaging the four vertices of the parallelogram to find the shift vector. We implement Step 4 according to [5]. From the rational matrix H computed in Step 4, Step 5 decomposes H into its Smith-McMillan form $H = U\,\Lambda\,V$, where Λ is a diagonal matrix with rational numbers along the diagonal and where U and V are integer matrices with determinant $+1$ or -1. Then, we collect terms:

$$H = U\,\Lambda\,V = U\,\Lambda_L^{-1}\,\Lambda_M\,V = \left(\Lambda_L\,U^{-1}\right)^{-1}(\Lambda_M V) = L^{-1}\,M \qquad (1)$$

So, $L = \Lambda_L\,U^{-1}$ and $M = \Lambda_M V$, where Λ_L and Λ_M are integer diagonal matrices. The diagonal elements of Λ_L are the reciprocals of the denominators of the diagonal elements of Λ, and the diagonal elements of Λ_M are the numerators of the diagonal elements of Λ. In the case of upsampling by $L = \Lambda_L\,U^{-1}$, input samples are first rearranged by U^{-1} (the rearrangement is lossless because U has a determinant of ± 1) and then separably upsampled by Λ_L (upsampled in the first dimension by Λ_{11} and in the second dimension by Λ_{22}). For downsampling by M, the input samples are first separably downsampled by Λ_M and then losslessly rearranged by V. L and M are relatively prime because the rational numbers along the diagonal elements of Λ are reduced by removing common factors before Λ_L and Λ_M are computed (for a definition of relative primeness, see [6, 7]). Because L and M are relatively prime, efficient polyphase implementations always exist for our rational decimator designs [6].

3 DESIGN EXAMPLES

This section describes the automatic design of a two-dimensional decimator for a sketched passband and a circular passband. The automated procedure is a realization of the steps outlined in the previous section by the routine `DesignDecimationSystem`. This routine takes one argument that is either a polygon or a rational sampling matrix. Given a rational sampling matrix, the routine just carries out the fifth step in the procedure from the previous section. Given a polygon, the routine performs all of the steps in the procedure. This routine supports an option `Mod` which sets an upper limit on the denominator of the coordinates of the parallelogram computed in step 2 of the procedure. The rest of this section discusses the two design examples.

In the first example, the user sketches the desired passband in the frequency domain with a mouse. Using two keystrokes, the user pastes the vertices of the polygon into the first command in Figure 2, and the second command draws the polygon. In the third command, the polygon is passed to the `DesignDecimationSystem` routine. The routine, because the `Dialogue` option has been set to `All`, plots the rectangle (in black) that circumscribes the passband and also plots the parallelogram (in black) that circumscribes the rectangle. For both plots, the original passband is shown in grey. The routine also reports the packing efficiency of the rectangle and the parallelogram, as well as the input-output compression ratio obtained by the decimation system. The compression ratio is defined as the area of the bounding parallelogram divided by the area of the fundamental frequency tile ($4\pi^2$), and it is computed by $|\det M|/|\det L|$. The routine returns the parameters of the compression system: the modulation shift $\mathbf{n}_0$, the upsampling matrix L, and the downsampling matrix M.

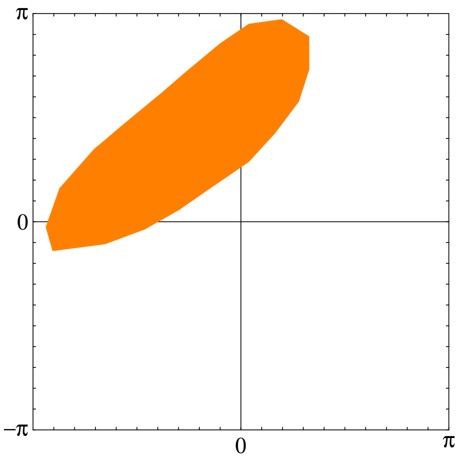
We ran the design procedure on a circular passband of radius 1. We approximated the circle with a twenty-sided polygon whose vertices were given by a simple *Mathematica* formula. The design procedure reported a packing efficiency of 79.2% and an 8-to-1 compression ratio. The best packing efficiency, 86.6%, is obtained by circumscribing the passband with a regular hexagon [1]. For non-linear compression, the theoretical upper limit on the compression ratio is 4π, which is approximately 12.5-to-1.

4 CONCLUSION

In some image, video, seismic applications, only a portion of the frequency content of the signals is important. This paper gives an automated procedure of the design of the the two-dimensional compression system shown in Figure 1 to resample the passband at its Nyquist rate. This approach is based on the knowledge of where the important passband resides in the frequency domain. The procedure circumscribes the passband with a parallelogram and maximally decimates the parallelogram.

```
poly = Polygon[
(* paste points below and evaluate expression *)
{{-2.846744, -0.443421}, {-2.950194, -0.081346},
{-2.743294, 0.50487}, {-2.226045, 1.091086},
{-1.777762, 1.470402}, {-1.191546, 1.953168},
{-0.81223, 2.280759}, {-0.312222, 2.694558},
{0.118819, 2.987666}, {0.618826, 3.056633},
{1.032626, 2.798008}, {1.032626, 2.298001},
{0.877451, 1.815234}, {0.515377, 1.332469},
{0.118819, 0.901428}, {-0.536364, 0.453145},
{-0.932921, 0.177279}, {-1.450171, -0.115829},
{-2.053628, -0.339971}}
];

Show [ Graphics[ { RGBColor[1,1/2,0], poly } ],
       AspectRatio -> 1, Axes -> True,
       Frame -> True,
       FrameTicks -> { piTicks, piTicks },
       PlotRange -> {{-Pi, Pi}, {-Pi, Pi}} ]
```



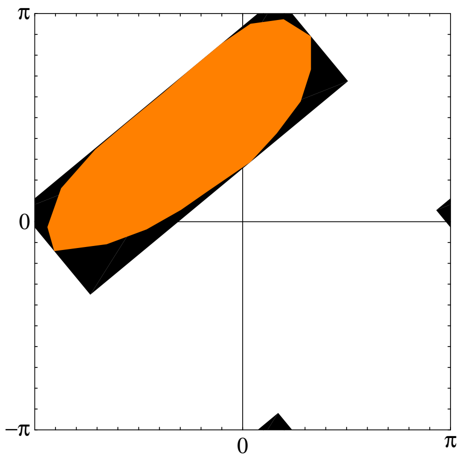

```
-Graphics-
```

This example finds the two-dimensional compression system that resamples the passband (shown directly above) at its Nyquist rate. The passband is sketched using a mouse. The software computes the rectangle that circumscribes the passband and the parallelogram with rational coordinates that circumscribes the rectangle. The parts of the circumscribing rectangle and parallelogram that extend outside of the fundamental frequency tile wrap around because the tile is periodic with period 2π in each frequency variable.

```
{ shift, upMatrix, downMatrix } =
  DesignDecimationSystem[
    poly, Dialogue -> All, Mod -> 10 ]
```

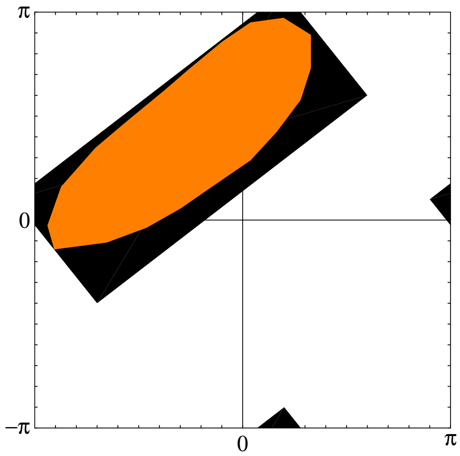
```
Best packing efficiency with rotated rectangle

   having real-valued coordinates: 78.7%
```




```
Actual packing efficiency: 63.6%

  (out of a best possible 78.7%)
```




```
The compression ratio is 80-to-21.
```

$\{\{\frac{-Pi}{4}, \frac{7\ Pi}{20}\}, \{\{21, -21\}, \{-4, 5\}\},$
$\{\{92, 4\}, \{-20, 0\}\}\}$

$$n_0 = (\frac{-\pi}{4}, \frac{7\pi}{20})$$

$$L = \begin{bmatrix} 21 & -21 \\ -4 & 5 \end{bmatrix}$$

$$M = \begin{bmatrix} 92 & 4 \\ -20 & 0 \end{bmatrix}$$

Figure 2: Automatic Design of a Decimator for an Arbitrarily-Shaped Passband

We have implemented the ideas in a set of signal processing packages and notebooks [3, 4] for the computer algebra system *Mathematica*. Our implementation allows the designer to specify the passband graphically using a mouse. For example, the designer could sketch the area of interest on top of a plot of the two-dimensional spectrum. Our implementation, however, does not design the two-dimensional filter. The decimation filter has a passband that is a parallelepiped symmetric about the origin whose vertices are $\pi L^{-1} M v_i$ for $i = 1 \ldots 4$ where $v = \{(-1, 1), (1, 1), (1, -1), (-1, -1)\}$ [5].

An area of future research could be the automation of the design of perfectly reconstructing two-dimensional filter banks [8, 9] based on an arbitrary geometric split of the fundamental frequency tile. Such an approach would utilize subband geometries that can be maximally decimated, i.e., parallelograms and regular hexagons.

References

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